PRESENTATIONS OF SEMIGROUP ALGEBRAS OF WEIGHTED TREES

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ABSTRACT. We find presentations for subalgebras of invariants of the coordinate algebras of binary symmetric models of phylogenetic trees studied by Buczynska and Wisniewski in [BW]. These algebras arise as toric degenerations of rings of global sections of weight varieties of the Grassmanian of two planes associated to the Plücker embedding, and as toric degenerations of rings of invariants of Cox-Nagata rings.

Contents

1. Introduction

Let $\mathcal{T}$ denote an abstract trivalent tree with leaves $V(\mathcal{T})$, edges $E(\mathcal{T})$, and non-leaf vertices $I(\mathcal{T})$, by trivalent we mean that the valence of v is three for any $v \in I(\mathcal{T})$. Let e_i be the unique edge incident to the leaf $i \in V(\mathcal{T})$. Let Y be the unique trivalent tree on three vertices. For each $v \in I(\mathcal{T})$ we pick an injective map $i_v : Y \to \mathcal{T}$, sending the unique member of $I(Y)$ to v. We denote the members of $E(Y)$ by E, F, and G.

Definition 1.1. *Let $S_\mathcal{T}$ be the graded semigroup where $S_\mathcal{T}[k]$ is the set of weightings*

$$\omega : E(\mathcal{T}) \to \mathbb{Z}_{\geq 0}$$

which satisfy the following conditions.

(1) *For all $v \in I(\mathcal{T})$ the numbers $i_v^*(\omega)(E)$, $i_v^*(\omega)(F)$ and $i_v^*(\omega)(G)$ satisfy*

$$|i_v^*(\omega)(E) - i_v^*(\omega)(F)| \leq i_v^*(\omega)(G) \leq |i_v^*(\omega)(E) + i_v^*(\omega)(F)|$$

These are referred to as the triangle inequalities.

(2) *$i_v^*(\omega)(E) + i_v^*(\omega)(F) + i_v^*(\omega)(G)$ is even.*

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(3) $\sum_{i \in V} \omega(e_i) = 2k$

In [SpSt] Speyer and Sturmfels show that the semigroup algebras $\mathbb{C}[S_{\mathcal{T}}]$ may be realized as rings of global sections for projective embeddings of the flat toric deformations of $Gr_2(\mathbb{C}^n)$ where $n = |V(\mathcal{T})|$, for the Plücker embedding. This semigroup is also multigraded, with the grading given by the weights $\omega(e_i)$.

Definition 1.2. *Let* $\mathbf{r} : V(\mathcal{T}) \to \mathbb{Z}_{\geq 0}$ *be a vector of nonnegative integers. Let* $S_{\mathcal{T}}(\mathbf{r})$ *be the multigraded subsemigroup of* $S_{\mathcal{T}}$ *formed by the pieces* $S_{\mathcal{T}}[k\mathbf{r}]$.

It follows from [SpSt] that graded algebras $\mathbb{C}[S_{\mathcal{T}}(\mathbf{r})]$ are rings of global sections for projective embeddings of flat toric deformations of $Gr_2(\mathbb{C}^n)//_{\mathbf{r}}T$, the weight variety of the Grassmanian of 2-planes associated to $\mathbf{r}$, or equivalently $\mathcal{M}_{\mathbf{r}}$, the moduli space of $\mathbf{r}$-weighted points on $\mathbb{C}P^1$. In [HMSV] this degeneration is used to construct presentations of the ring of global sections for a projective embedding of $\mathcal{M}_{\mathbf{r}}$, and it was shown for certain $\mathbf{r}$ and $\mathcal{T}$ that these algebras are generated in degree 1 and have relations generated by quadrics and cubics. This is the starting point for the present paper, along with the work of Buczynska and Wisniewski [BW], where it was shown that the algebras associated to the following semigroups all have the same multigraded Hilbert function, with the multigrading defined as it is for $S_{\mathcal{T}}$.

Definition 1.3. *Let* L *be a positive integer. Let* $S_{\mathcal{T}}^L$ *be the graded semigroup where* $S_{\mathcal{T}}^L[k]$ *is the set of weightings* ω *of* $\mathcal{T}$ *which satisfies the same conditions as* $S_{\mathcal{T}}[k]$ *with the addition assumption that*

$$i_v^*(\omega)(E) + i_v^*(\omega)(F) + i_v^*(\omega)(G) \leq 2kL.$$

This is referred to as the level condition.

In [StX], Sturmfels and Xu construct the multigraded Cox-Nagata ring $R^G(L)$, which functions as an analogue of $Gr_2(\mathbb{C}^n)$ in that it can be flatly deformed to each $\mathbb{C}[S_{\mathcal{T}}^L]$. The analogue of the weight varieties in this context are the multigrade $\mathbf{r}$ Veronese subrings of $R^G(L)$, denoted $R^G(L)_{\mathbf{r}}$.

Definition 1.4. *Let* L *be a positive integer. Let* $S_{\mathcal{T}}^L(\mathbf{r})$ *be the multigraded subsemigroup of* $S_{\mathcal{T}}^L$ *of summands with multigrade* $k\mathbf{r}$.

Remark 1.5. *The multigraded Hilbert functions of* $\mathbb{C}[S_{\mathcal{T}}^L]$ *and* $\mathbb{C}[S_{\mathcal{T}}^L(\mathbf{r})]$ *are closely related to the Verlinde Formula for* $SU(2)$ *(see* [BW] *and* [StX]*). Indeed, the reader may notice that the defining conditions for* $S_{\mathcal{T}}^L$ *and* $S_{\mathcal{T}}^L(\mathbf{r})$ *are given by Quantum Clebsch-Gordon Rules for* $SU(2)$, *whereas the defining conditions for* $S_{\mathcal{T}}$ *and* $S_{\mathcal{T}}(\mathbf{r})$ *are classical* $SU(2)$ *Clebsch-Gordon Rules.*

It follows from results in [StX] that $\mathbb{C}[S_{\mathcal{T}}^L(\mathbf{r})]$ is a toric deformation of $R^G(L)_{\mathbf{r}}$. In this paper we construct presentations for a large class of the rings $\mathbb{C}[S_{\mathcal{T}}^L(\mathbf{r})]$. The techniques used are such that the same results immediately hold for $\mathbb{C}[S_{\mathcal{T}}(\mathbf{r})]$ as well, in particular we give a different proof of a fundamental result of [HMSV] on a presentation of these rings.

1.1. **Statement of Results.** We now state the main results of the paper. When two leaves are both connected to a common vertex, we say they are paired to each other. A leaf that has no pair is called a lone leaf.

Definition 1.6. *We call the triple* $(\mathcal{T}, \mathbf{r}, L)$ *admissible if* L *is even,* $\mathbf{r}(i)$ *is even for every lone leaf* i, *and* $\mathbf{r}(j) + \mathbf{r}(k)$ *is even for all paired leaves* j, k.

Remark 1.7. *Admissability is actually not very restrictive. Note that the assumption that* $\mathbf{r}$ *has an even total sum gurantees that we may find a* $\mathcal{T}$ *such that* $(\mathcal{T}, \mathbf{r}, L)$ *is admissible for any even* L*. This is important for constructing presentations of* $R^G(L)_{\mathbf{r}}^T$*, since this ring always has a flat deformation to* $\mathbb{C}[S_{\mathcal{T}}^L(\mathbf{r})]$ *for some admissible* $(\mathcal{T}, \mathbf{r}, L)$*. Also note that the second Veronese subring of* $\mathbb{C}[S_{\mathcal{T}}^L(\mathbf{r})]$ *is the semigroup algebra associated to* $(\mathcal{T}, 2\mathbf{r}, 2L)$*, which is always admissible.*

Theorem 1.8. *For* $(\mathcal{T}, \mathbf{r}, L)$ *admissible with* $L > 2$*,* $\mathbb{C}[S_{\mathcal{T}}^L(\mathbf{r})]$ *is generated in degree* 1.

Theorem 1.9. *For* $(\mathcal{T}, \mathbf{r}, L)$ *admissible with* $L > 2$*,* $\mathbb{C}[S_{\mathcal{T}}^L(\mathbf{r})]$ *has relations generated in degree at most* 3*.*

As a corollary we get the same results for $S_{\mathcal{T}}(\mathbf{r})$ when $(\mathcal{T}, \mathbf{r})$ satisfy admissibility conditions. These theorems will be proved in sections 2, 3, and 4. In section 5 we will look at some special cases, and investigate what can go wrong when $(\mathcal{T}, \mathbf{r}, L)$ is not an admissible triple.

1.2. **Outline of techniques.** To prove Theorems 1.8 and 1.9 we use two main ideas. First, we employ the following trivial but useful observation.

Proposition 1.10. *Let* $(\mathcal{T}, \mathbf{r}, L)$ *be admissible, then for any weighting* $\omega \in S_{\mathcal{T}}^L(\mathbf{r})$*,* $\omega(e)$ *is an even number when* e *is not an edge connected to a paired leaf.*

This allows us to drop the parity condition that $i_v^*(\omega)(E) + i_v^*(\omega)(F) + i_v^*(\omega)(G)$ is even by forgetting the paired leaves and halving all remaining weights.

Definition 1.11. *Let* $c(\mathcal{T})$ *be the subtree of* $\mathcal{T}$ *given by forgetting all edges incident to paired leaves.*
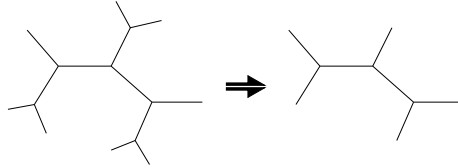

FIGURE 1. Clipping the paired leaves of $\mathcal{T}$

Definition 1.12. *Let* $U_{c(\mathcal{T})}^L(\mathbf{r})$ *be the graded semigroup of weightings on* $c(\mathcal{T})$ *such that the members of* $U_{c(\mathcal{T})}^L(\mathbf{r})[k]$ *satisfy the triangle inequalities, the new level condition* $i_v^*(\omega)(E) + i_v^*(\omega)(F) + i_v^*(\omega)(G) \leq L$*, and the following conditions.*

(1) $\omega(e_i) = k\frac{\mathbf{r}(i)}{2}$ *for* i *a lone leaf of* $\mathcal{T}$*.*
(2) $\frac{k|\mathbf{r}(j) - \mathbf{r}(k)|}{2} \leq \omega(e) \leq \frac{k|\mathbf{r}(j) + \mathbf{r}(k)|}{2}$ *for* e *the unique edge of* $\mathcal{T}$ *connected to the vertex which is connected to the paired leaves* i *and* j*.*

Also, let $U_{c(\mathcal{T})}^L$ be the graded semigroup of weightings which satisfy the triangle inequalities and the new level condition for L. The following is a consequence of these definitions.

Proposition 1.13. *For $(\mathcal{T}, \mathbf{r}, L)$ admissible,*

$$U^L_{c(\mathcal{T})}(\mathbf{r}) \cong S^L_{\mathcal{T}}(\mathbf{r})$$

as graded semigroups.

We refer to the graded semigroup of weightings on Y which satisfy the triangle inequalities and the new L level condition as U^L_Y. The next main idea is to undertake the analysis of $U^L_{c(\mathcal{T})}(\mathbf{r})$ by first considering the weightings $i^*_v(\omega) \in U^L_Y$. After constructing a pertinant object in U^L_Y, like a factorization or relation, we "glue" these objects back together along edges shared by the various $i_v(Y)$ with what amounts to a fibered product of graded semigroups. This is remniscent of the theory of moduli of orientable surfaces, where structures on a surface of high genus can be glued together from structures on three-punctured spheres over a pair-of-pants decomposition. The reason for this resemblance is not entirely accidental, see [HMM]. We obtain information about U^L_Y by studying the following polytope.

Remark 1.14. *In* [BW], *Buczynska and Wisniewski used more or less the same idea. They prove facts about $S^L_{\mathcal{T}}$ by viewing it as a fibered product of copies of S^L_Y.*

Definition 1.15. *Let $P_3(L)$ be the convex hull of $(0,0,0)$, $(\frac{L}{2}, \frac{L}{2}, 0)$, $(\frac{L}{2}, 0, \frac{L}{2})$, and $(0, \frac{L}{2}, \frac{L}{2})$.*
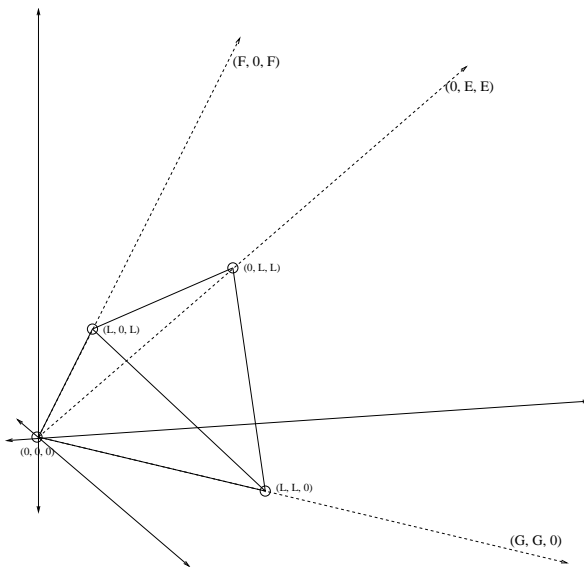

FIGURE 2. $P_3(2L)$

The graded semigroups of lattice points for $P_3(L)$ is U^L_Y. By a lattice equivalence of polytopes $P, Q \subset \mathbb{R}^n$ with respect to a lattice $\Lambda \subset \mathbb{R}^n$ we mean a composition of translations by members of Λ and members of $GL(\Lambda) \subset GL_n(\mathbb{R})$ which takes P to Q. If P and Q are lattice equivalent it is easy to show that they have isomorphic graded semigroups of lattice points. When L is an even integer (admissibility condition) the interesection of this polytope with any translate of the unit cube in $\mathbb{R}^3$, is, up to lattice equivalence, one of the polytopes shown in figure 3.
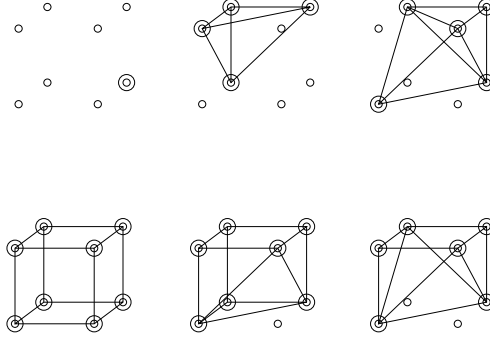
FIGURE 3. Cube Polytopes

Each of these polytopes is normal, and the relations of their associated semi-groups are generated in degree at most 3. In sections 3 and 4 we will lift these properties to $U^L_{c(\mathcal{T})}(\mathbf{r})$, and therefore $S^L_{\mathcal{T}}(\mathbf{r})$ for $(\mathcal{T}, \mathbf{r}, L)$ admissible. Facts about the six polytopes above also allow us to carry out a more detailed investigation into the properties of the semigroups $S^L_{\mathcal{T}}(\mathbf{r})$ in section 5, for example they allow us to show the redundancy of the cubic relations for certain $(\mathcal{T}, \mathbf{r}, L)$.

I would like to thank John Millson for introducing me to this problem, Ben Howard for many useful and encouraging conversations and for first introducing me to the commutative algebra of semigroup rings, Larry O'Neil for several useful conversations on the cone of triples which satisfy the triangle inequalities, and Bernd Sturmfels for introducing me to Graver bases and shortening the proof of Theorems and 2.2 and 2.4.

2. The Cube Semigroups

In this section we will prove that the intersection of any translate of the unit cube of $\mathbb{R}^3$ with $P_3(2L)$ produces a normal polytope whose semigroup of lattice points has relations generated in degree at most 3 when L is an integer. Let P_3 be the cone of triples of nonnegative integers which satisfy the triangle inequalities, and let

$$C(m_1, m_2, m_3) = conv\{(m_1 + \epsilon_1, m_2 + \epsilon_2, m_3 + \epsilon_3) | \epsilon_i \in \{0, 1\}\}$$

We want classify the polytopes which have the presentation $C(m_1, m_2, m_3) \cap P_3$, since P_3 is symmetric we may assume that (m_1, m_2, m_3) is ordered by magnitude with m_3 the largest. We will keep track of the triangle inequalities with the quantities $n_i = m_j + m_k - m_i$. For a point (m_1, m_2, m_3) to be in P_3 is equivalent to $n_i \geq 0$ for each i. Immediately we have the following inequalities.

$$n_1 \geq n_2 \geq n_3, n_2 \geq 0$$

If $n_3 < -2$ then no member of $C(m_1, m_2, m_3)$ can belong to P_3. If $n_3 \geq -2$ then there are six distinct possibilities, we list each case along with the members of $C(m_1, m_2, m_3) \cap P_3 - (m_1, m_2, m_3)$.

$Condition$	$C(m_1, m_2, m_3) \cap P_3 - (m_1, m_2, m_3)$
$n_3 = -2$	$(1, 1, 0)$
$n_3 = -1$	$(1, 1, 0), (0, 1, 0), (1, 0, 0), (1, 1, 1)$
$n_1 = n_2 = n_3 = 0$	$(1, 1, 0), (0, 1, 1), (1, 0, 1), (1, 1, 1), (0, 0, 0)$
$n_1 > 0, n_2 = n_3 = 0$	$(1, 1, 0), (0, 1, 1), (1, 0, 1), (1, 1, 1), (0, 0, 0), (0, 0, 1)$
$n_1, n_2 > 0, n_3 = 0$	$(1, 1, 0), (0, 1, 1), (1, 0, 1), (1, 1, 1), (0, 0, 0), (0, 0, 1), (0, 1, 0)$
$n_i > 0$	all points

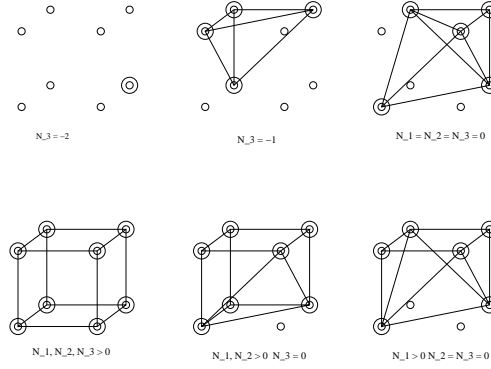The figure below illustrates these arrangements.

FIGURE 4.   Primitive cube semigroups

Now we will see what happens when we intersect P_3 with the half plane $m_1 + m_2 + m_3 \leq 2L$ to get $P_3(2L)$. The reader may want to refer to figure 5 for this part. The convex set $C(m_1, m_2, m_3) \cap P_3(2L)$ can be one of the above polytopes (up to lattice equivalence), or one of them intersected with the half plane $m_1 + m_2 + m_3 \leq 2L$. Note that a vertex v in $C(m_1, m_2, m_3) \cap P_3(2L)$ lying on a facet of P_3 necessarily satisfies $v_1 + v_2 + v_3 = 0 \ mod \ 2$. In Figure 5 these points are colored black.
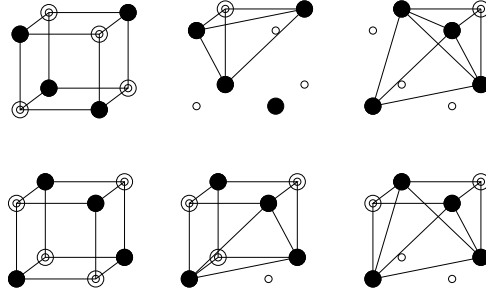

FIGURE 5. Cube semigroups with the lattice $v_1 + v_2 + v_3 = 0 \ mod \ 2$

The hyperplane $m_1 + m_2 + m_3 = 2L$ must intersect these polytopes at collections of three black points. If we assume that the lower left corner is $(0, 0, 0)$, these points have coordinates $\{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$, or $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. Figure 5 represents the new possibilities for $C(m_1, m_1, m_3) \cap P_3(2L) - (m_1, m_2, m_3)$. The polytope pictured lower center in Figure 6 is the only case which is not lattice equivalent to one pictured in Figure 4. It is rooted at $(0, 0, 0)$ and occurs only

when $L = 1$ (level condition is 2). The point $(1, 1, 1)$ in its second Minkowski sum cannot be expressed as the sum of two lattice points of degree one, so this is not a normal polytope. This is the reason we stipulate that $L > 2$ in Theorem 1.8. Now we analyze each $C(m_1, m_2, m_3) \cap P_3(2L)$. Since lattice equivalent polytopes have isomorphic semigroups of lattice points, it suffices to investigate the polytopes listed in Figure 4.
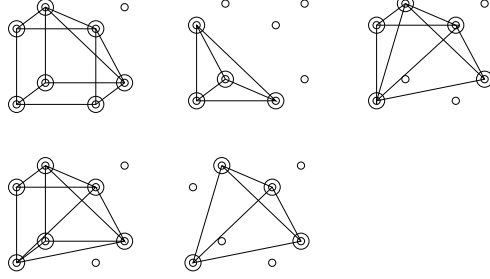

FIGURE 6. New Possibilities for $C(m_1, m_2, m_3) \cap P_3(2L)$

Caution 2.1. *In [BW], Buczynska and Wisniewski study a normal polytope with the same vertices as the non-normal polytope mentioned above. This is possible because they are using the the lattice $v_1 + v_2 + v_3 = 0 \ mod \ 2$, not the standard lattice.*

We make use of the computational algebra package 4ti2, [4ti2] to compute the *Graver basis* of the toric ideal of the unit 3-cube.

$$(1,0,0) + (1,1,1) = (1,0,1) + (1,1,0) \qquad (0,1,0) + (1,1,1) = (0,1,1) + (1,1,0)$$
$$(0,0,0) + (1,1,1) = (0,0,1) + (1,1,0) \qquad (0,0,1) + (1,1,1) = (0,0,1) + (1,0,1)$$
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$$(0,0,1) + (0,1,0) + (1,1,1) = (0,1,1) + (0,1,1) + (1,0,0)$$
$$(0,0,0) + (0,0,0) + (1,1,1) = (1,0,0) + (0,1,0) + (0,0,1)$$

Operating on this set of monomials, one can show that the toric ideal of every sub-polytope of the unit 3-cube which is not a simplex has a square-free Gröbner basis. This, combined with the fact that the sub-polytopes with $n_3 = -2$ and -1 are unimodular simplices shows the following theorem.

Theorem 2.2. *Let $L \neq 1$, then for all (m_1, m_2, m_3), if $C(m_1, m_2, m_3) \cap P_3(2L)$ is non-empty, then it is normal.*

Remark 2.3. This theorem implies, among other things, that if $\omega \in U_Y(2L)[k]$, then

$$\omega = \sum_{i=1}^{k} W_i$$

for $W_i \in P_3(2L)$ with the property that each

$$W_i = X + (\epsilon_1, \epsilon_2, \epsilon_3)$$

with $\epsilon_j \in \{0, 1\}$ for all i for a fixed $X \in \mathbb{R}^3$. It is easy to show that

$$X = (\lfloor \frac{\omega(E)}{k} \rfloor, \lfloor \frac{\omega(F)}{k} \rfloor, \lfloor \frac{\omega(G)}{k} \rfloor)$$

Therefore each W_i is $(\frac{\omega(E)}{k}, \frac{\omega(F)}{k}, \frac{\omega(G)}{k})$ with either floor or ceiling applied to each entry.

Now we move on to relations, Let $S(m_1, m_2, m_3)$ be the semigroup of lattice points for $C(m_1, m_2, m_3) \cap P_3(2L) - (m_1, m_2, m_3)$, once again it suffices to treat the cases represented in Figure 4.

Theorem 2.4. *All relations for the semigroup $S(m_1, m_2, m_3)$ are reducible to quadrics and cubics.*

Proof. By Proposition 4.13 of [St], a Graver basis for any subpolytope P of the unit 3-cube is obtained by taking the members of the Graver basis of the unit 3-cube which have entries in the lattice points of P. Since these are all quadrics and cubics, we are done. □

Up to equivalence and after accounting for redundancy, all relations are of the form

$$(1, 0, 0) + (0, 1, 0) = (1, 1, 0) + (0, 0, 0)$$
$$(1, 0, 1) + (0, 1, 0) = (1, 1, 1) + (0, 0, 0)$$
$$(1, 0, 1) + (1, 1, 0) = (1, 1, 1) + (1, 0, 0)$$
$$(1, 1, 1) + (1, 1, 1) + (0, 0, 0) = (1, 1, 0) + (1, 0, 1) + (0, 1, 1),$$

with the last one the only degree 3 relation, we refer to it as the "degenerated Segre Cubic" (see [HMSV]).

3. PROOF OF THEOREM 1.8

In this section we use Theorem 2.2 to prove that $U_{c(\mathcal{T})}^L(\mathbf{r})$ is generated in degree 1, which then proves Theorem 1.8. For each $v \in I(\mathcal{T})$ we have the morphism of graded semigroups

$$i_v^* : U_{c(\mathcal{T})}^L(\mathbf{r}) \to U_Y^L.$$

Given a weight $\omega \in U_{c(\mathcal{T})}^L(\mathbf{r})$ we factor $i_v^*(\omega)$ for each $Y \subset c(\mathcal{T})$ using Theorem 2.2. Then, special properties of the weightings obtained by this procedure will allow us to glue the factors of the $i_v^*(\omega)$ back together along common edges to obtain a factorization of ω. First we must make sure that our factorization procedure does not disrupt the conditions at the edges of $c(\mathcal{T})$.

Lemma 3.1. *Let $\omega \in U_{c(\mathcal{T})}^L(\mathbf{r})[k]$, and let $v \in I(\mathcal{T})$ be connected to a leaf of $c(\mathcal{T})$, at E. Then if $i_v^*(\omega) = \eta_1 + \ldots + \eta_k$ is any factorization of $i_v^*(\omega)$ with $\eta_i \in C(\lfloor \frac{i_v^*(\omega)(E)}{k} \rfloor, \lfloor \frac{i_v^*(\omega)(F)}{k} \rfloor, \lfloor \frac{i_v^*(\omega)(G)}{k} \rfloor)$ Then $\eta_i(E)$ satisfies the appropriate edge condition for elements in $U_{c(\mathcal{T})}^L(\mathbf{r})[1]$.*

Proof. If E is attached to a lone leaf of $\mathcal{T}$ then $i_v^*(\omega)(E) = k\mathbf{r}(e)$ for $i_v(E) = e$, $e \in V(\mathcal{T})$. By Remark 2.3

$$\eta_i(E) = \lfloor \mathbf{r}(e) \rfloor = \mathbf{r}(e)$$

or

$$\eta_i(E) = \lfloor \mathbf{r}(e) \rfloor + 1 = \mathbf{r}(e) + 1$$

Since $\sum_{i=1}^k \eta_i(E) = k\mathbf{r}(e)$ we must have $\eta_i(E) = \mathbf{r}(e)$ for all i. If E is a stalk of paired leaves i and j in $\mathcal{T}$ then we must have

$$k \frac{|\mathbf{r}(i) - \mathbf{r}(j)|}{2} \leq \omega_Y(E) \leq k \frac{|\mathbf{r}(i) + \mathbf{r}(j)|}{2}$$

Note that both bounds are divisible by k. Since floor preserves lower bounds we have

$$\frac{|\mathbf{r}(i) - \mathbf{r}(j)|}{2} \leq \lfloor \frac{i_v^*(\omega)(E)}{k} \rfloor,$$

and since ceiling preserves upper bounds we have

$$\lceil \frac{i_v^*(\omega)(E)}{k} \rceil \leq \frac{|\mathbf{r}(i) + \mathbf{r}(j)|}{2}.$$

Therefore each η_i satisfies

$$\frac{|\mathbf{r}(i) - \mathbf{r}(j)|}{2} \leq \eta_i(E) \leq \frac{|\mathbf{r}(i) + \mathbf{r}(j)|}{2}$$

$\square$

Now that we can safely use Theorem 2.2 with each $i_v^* : U_{c(\mathcal{T})}^L(\mathbf{r}) \to U_Y^L$, we can see about gluing these factors together along common edges.

Definition 3.2. *We say a set of nonnegative integers $\{X_1, \ldots, X_n\}$ is balanced if $|X_i - X_j| = 1$ or 0 for all i, j.*

The following is a very useful lemma, its proof is left to the reader.

Lemma 3.3. *If two sets $\{X_1, \ldots, X_n\}$ and $\{Y_1, \ldots, Y_m\}$ are balanced, have the same total sum, and $n = m$, then they are the same set.*

Proposition 3.4. *The semigroup $U_{c(\mathcal{T})}^L(\mathbf{r})$ is generated in degree 1.*

Proof. Recall that by Remark 2.3, for any edge $E \in Y$ the edge weights of a factorization $i_v^*(\omega) = \eta_1 + \ldots \eta_k$ satisfy $\eta_i(E) = \lfloor \frac{i_v^*(\omega)(E)}{k} \rfloor$ or $\lceil \frac{i_v^*(\omega)(E)}{k} \rceil$. Take any two v_1, v_2 which share a common edge E in $c(\mathcal{T})$. Let $\omega \in U_{c(\mathcal{T})}^L(\mathbf{r})[k]$ and let $\{\eta_1^1, \ldots, \eta_k^1\}$ and $\{\eta_1^2, \ldots, \eta_k^2\}$ be factorizations of $i_{v_1}^*(\omega)$ and $i_{v_1}^*(\omega)$ respectively. Then the sets $\{\eta_1^1(E), \ldots, \eta_k^1(E)\}$ and $\{\eta_1^2(E), \ldots, \eta_k^2(E)\}$ are balanced and have the same sum, so by Lemma 3.3 they are the same set. We may glue factors η_i^1 and η_j^2 when $\eta_i^1(E) = \eta_j^2(E)$, the above observation gurantees that any η_i^1 has an

available partner η_j^2. The proposition now follows by induction on the number of $v \in I(c(\mathcal{T}))$. This implies Theorem 1.8. $\qquad\square$

4. Proof of Theorem 1.9

In this section we show how to get all relations in $U_{c(\mathcal{T})}^L(\mathbf{r})$ from those lifted from U_Y^L. The procedure follows the same pattern as the proof of Theorem 1.8. We consider the image of a relation $\omega_1 + \ldots + \omega_n = \eta_1 + \ldots + \eta_n$ under a map $i_v^* : U_{c(\mathcal{T})}^L(\mathbf{r}) \to U_Y^L$, using Theorem 2.4 we convert this to a trivial relation using relations of degree at most 3. We then give a recipe for lifting each of these relalations back to $U_{c(\mathcal{T})}^L(\mathbf{r})$. The result is a way to convert $\omega_1 + \ldots + \omega_n = \eta_1 + \ldots + \eta_n$ to a relation which is trivial over the trinode v using quadrics and cubics. In this way we take a general relation to a trivial relation one $v \in I(c(\mathcal{T}))$ at a time.

Definition 4.1. *A set of degree 1 elements $\{\omega_1, \ldots, \omega_k\}$ in $U_{c(\mathcal{T})}^L(\mathbf{r})$ is called Balanced when the set $\{\omega_1(E), \ldots, \omega_k(E)\}$ is balanced for all $E \in c(\mathcal{T})$. A relation $\omega_1 + \ldots + \omega_k = \eta_1 + \ldots + \eta_k$ in $U_{c(\mathcal{T})}^L(\mathbf{r})$ is called Balanced when $\{\omega_1, \ldots, \omega_k\}$ and $\{\eta_1, \ldots, \eta_k\}$ are balanced.*

The following lemmas say that we need only consider balanced relations.

Lemma 4.2. *Any set of nonnegative integers $S = \{X_1, \ldots, X_n\}$ can be converted to a balanced set $T = \{Y_1, \ldots, Y_n\}$ with $\sum_{i=1}^n Y_i = \sum_{i=1}^n X_i$ by replacing a pair X_i and X_j with $\lfloor \frac{X_i + X_j}{2} \rfloor$ and $\lceil \frac{X_i + X_j}{2} \rceil$ a finite number of times.*

Proof. Let $d(S)$ be the difference between the maximum and mininum elements of S. It is clear that with a finite number of exchanges

$$\{X_i, X_j\} \to \{\lfloor \frac{X_i + X_j}{2} \rfloor, \lceil \frac{X_i + X_j}{2} \rceil\}$$

We get a new set S' with $d(S) > d(S')$, unless $d(S) = 1$ or 0. Since this happens if and only of S is balanced, the lemma follows by induction. $\qquad\square$

Lemma 4.3. *Let*

$$\omega_1 + \ldots + \omega_k = \eta_1 + \ldots + \eta_k$$

be a relation in $U_{c(\mathcal{T})}^L(\mathbf{r})$ then it can be converted to a balanced relation

$$\omega_1' + \ldots + \omega_k' = \eta_1' + \ldots + \eta_k'$$

using only degree 2 relations.

Proof. First we note that using the proof of Theorem 1.8 we can factor the weighting $\omega_1 + \omega_2$ into $\omega_1' + \omega_2'$ so that $\{\omega_1', \omega_2'\}$ is balanced. Using this and Lemma 4.2 we can find

$$\omega_1' + \ldots + \omega_k' = \omega_1 + \ldots + \omega_k$$

such that the set $\{\omega_1'(E), \ldots, \omega_k'(E)\}$ is balanced for some specific E, using only degree 2 relations. Observe that if $\{\omega_1(F), \ldots, \omega_k(F)\}$ is balanced for some F, the same is true for $\{\omega_1'(F), \ldots, \omega_k'(F)\}$, after a series of degree 2 applications of 1.8 as above. This shows that we may inductively convert $\{\omega_1, \ldots, \omega_k\}$ to $\{\omega_1', \ldots, \omega_k'\}$ with the property that $\{\omega_1'(E), \ldots, \omega_k'(E)\}$ is a balanced set for all edges E, using only degree 2 relations. Applying the same procedure to the η_i then proves the lemma. $\qquad\square$

The next lemma shows how we lift a balanced relation in U_Y^L to one in $U_{c(\mathcal{T})}^L(\mathbf{r})$.

Lemma 4.4. *Let $\{\omega_1 \ldots \omega_k\}$ be a balanced set of elements in $U_{c(\mathcal{T})}^L(\mathbf{r})$. Let $i_v^*(\omega_1) + \ldots + i_v^*(\omega_k) = \eta_1 + \ldots + \eta_k$ be a degree k relation the appropriate $S(m_1, m_2, m_3) \subset U_Y^L$. Then the η_i may be lifted to weightings of $c(\mathcal{T})$ giving a relation of degree k in $U_{c(\mathcal{T})}^L(\mathbf{r})$ which agrees with the relation above when i_v^* is applied, and is a permutation of $i_{v'}^*(\omega_1) \ldots i_{v'}^*(\omega_N)$ for $v' \neq v$.*

Proof. Let $c(\mathcal{T})(E)$ be the unique connected subtrivalent tree of $c(\mathcal{T})$ which includes v and has the property that any path $\gamma \subset c(\mathcal{T})(E)$ with endpoints at a vertex $v' \neq v$ in $c(\mathcal{T})(E)$ and v includes the edge E (see Figure 7), define $c(\mathcal{T})(F)$ and $c(\mathcal{T})(G)$ in the same way. To make $\eta_1' \ldots \eta_k'$ over $c(\mathcal{T})$, note that the set $\{i_{c(\mathcal{T})(E)}^*(\omega_i)(E)\}$ is the same as the set $\{\eta_i(E)\}$, because they are both balanced sets with the same sum and the same number of elements, so we may glue these weightings together to make a tuple over $c(\mathcal{T})$. $\qquad\square$
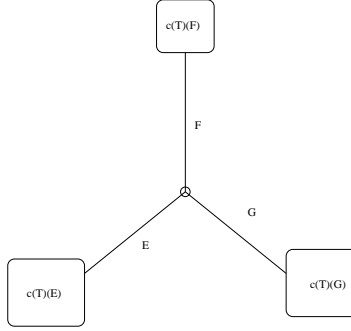

FIGURE 7. Component subtrees about a vertex

If we are given a relation

$$\omega_1 + \ldots + \omega_k = \eta_1 + \ldots + \eta_k$$

with both sides balanced, we may use relations in the appropriate $S(m_1, m_2, m_3)$ to convert $\{\omega_1, \ldots, \omega_k\}$ to $\{\eta_1, \ldots, \eta_k\}$ one $v \in I(c(\mathcal{T}))$ at a time. This leads us to the following proposition.

Proposition 4.5. *Let N be the maximum degree of relations needed to generate all relations in the semigroups $S(m_1, m_2, m_3)$. Then the semigroup $U_{c(\mathcal{T})}^L(\mathbf{r})$ has relations generated in degree bounded by N.*

This proposition, coupled with Theorem 2.4 proves Theorem 1.9. We recap the content of the last two sections with the following theorem.

Theorem 4.6. *Let $(\mathcal{T}, \mathbf{r}, L)$ be admissible. Then the ring $\mathbb{C}[U_{c(\mathcal{T})}^L(\mathbf{r})]$ has a presentation*

$$0 \longrightarrow I \longrightarrow \mathbb{C}[X] \longrightarrow \mathbb{C}[U_{c(\mathcal{T})}^L(\mathbf{r})] \longrightarrow 0$$

where X is the set of degree 1 elements of $U_{c(\mathcal{T})}^L(\mathbf{r})$, and I is the ideal generated by two types of binomials,

$$[\omega_1] \circ \ldots \circ [\omega_n] - [\eta_1] \circ \ldots \circ [\eta_n].$$

(1) *Binomials where $n \leq 3$, $i_v^*(\omega_1) + \ldots + i_v^*(\omega_n) = i_v^*(\eta_1) + \ldots + i_v^*(\eta_n)$ is a balanced relation in U_Y^L for some specific v, and $\{i_{v'}^*(\omega_1), \ldots, i_{v'}^*(\omega_n)\} = \{i_{v'}^*(\eta_1), \ldots, i_{v'}^*(\eta_n)\}$ for $v \neq v'$.*
(2) *Binomials where $n = 2$ and $i_v^*(\omega_1) + i_v^*(\omega_2) = i_v^*(\eta_1) + i_v^*(\eta_2)$ such that $\{i_v^*(\omega_1), i_v^*(\omega_2)\}$ is balanced for all $v \in I(c(\mathcal{T}))$.*

This induces a presentation for $\mathbb{C}[S_{\mathcal{T}}^L(\mathbf{r})]$ by isomorphism.

Corollary 4.7. *The same holds for $\mathbb{C}[S_{\mathcal{T}}(\mathbf{r})]$.*

Proof. For each pair $(\mathcal{T}, \mathbf{r})$ it is easy to show that there is a number $N(\mathcal{T}, \mathbf{r})$, such that any weighting ω which satisfies the triangle inequalities on $\mathcal{T}$ and has $\omega(e_i) = \mathbf{r}_i$ must have $\omega(e) \leq N(\mathcal{T}, \mathbf{r})$ for $e \in E(\mathcal{T})$. Because of this $S_{\mathcal{T}}^L(\mathbf{r}) = S_{\mathcal{T}}(\mathbf{r})$ for L sufficiently large. $\square$

5. Special Cases and Observations

In this section we collect results on some special cases of $\mathbb{C}[S_{\mathcal{T}}^L(\mathbf{r})]$. In particular we study some instances when cubic relations are unnecessary, we give some examples where the semigroup is not generated in degree 1, we analyze the case when L is allowed to be odd, and we give instances where cubic relations are necessary.

5.1. **The Caterpillar Tree.** One consequence of the proof of Theorem 2.4 is that a semigroup $U_{c(\mathcal{T})}^{2L}(\mathbf{r})$ which omits or only partially admits the semigroup $S(0, 0, 0)$ or $S(L-1, L-1, 0)$ as an image of one of the morphisms i_v^* manages to avoid degree 3 relations entirely. The next proposition illustrates one such example, the semigroups of weightings on the Caterpillar tree, pictured below.
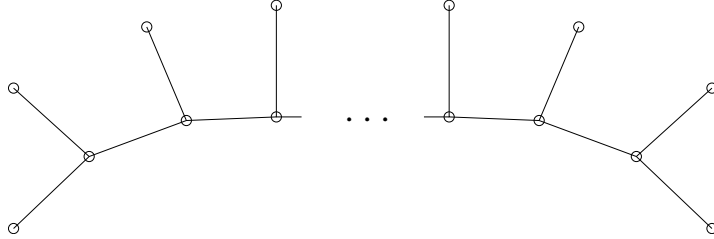

FIGURE 8. The Caterpillar tree

Proposition 5.1. *Let $\mathcal{T}_0$ be the caterpillar tree, and let $\mathbf{r}(i)$ be even for all $i \in V(\mathcal{T}_0)$, then $S_{\mathcal{T}_0}^{2L}(\mathbf{r})$ is generated in degree 1, with relations generated by quadrics.*

Proof. We catalogue the weights $i_v^*(\omega)$ which can appear in degree 1. For the sake of simplicity we divide all weights by 2. Suppose $i_v(G)$ is an external edge, then $i_v^*(\omega)(E)$ and $i_v^*(\omega)(F)$ satisfy the following inequalities

$$i_v^*(\omega)(E) \leq i_v^*(\omega)(F) + \frac{\mathbf{r}(i)}{2}$$

$$i_v^*(\omega)(F) \leq i_v^*(\omega)(E) + \frac{\mathbf{r}(i)}{2}$$

$$i_v^*(\omega)(E) + i_v^*(\omega)(F) + \frac{\mathbf{r}(i)}{2} \leq 2L$$

where $i_v^*(\omega)(G) = \mathbf{r}(i)$. These conditions define a polytope in $\mathbb{R}^2$ with vertices $(L, L - \frac{\mathbf{r}(i)}{2})$, $(L - \frac{\mathbf{r}(i)}{2}, L)$, $(\frac{\mathbf{r}(i)}{2}, 0)$ and $(0, \frac{\mathbf{r}(i)}{2})$. Pictured below is the case $L = 9$, $\mathbf{r}(i) = 6$. When two edges are external, the polytope is an integral line segment.
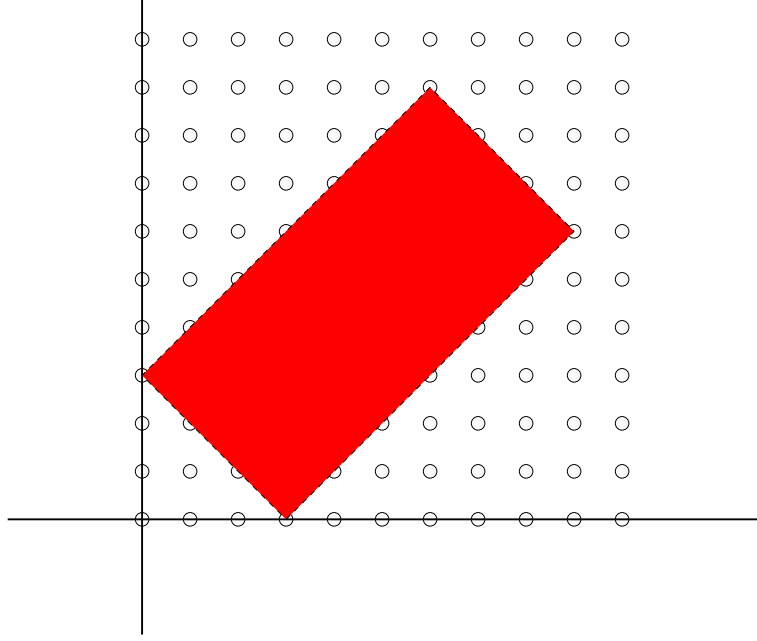

FIGURE 9. The case $L = 9$, $\mathbf{r}(i) = 6$

Note that the intersection of any lattice cube in $\mathbb{R}^2$ with the above polytope is a simplex or a unit square. Both of these polytopes have at most quadrics for relations in their semigroup of lattice points. Hence the argument used to prove Theorem 1.9 shows that $U_{c(\mathcal{T}_0)}^{2L}(\mathbf{r})$ needs only quadric relations. $\square$

Corollary 5.2. *If L is even and greater than 2, and $\mathbf{r}$ is a vector of nonnegative even integers, the ring $R^G(L)_\mathbf{r}$ has a presentation with defining ideal generated by quadrics. In particular, the second Veronese subring of any $R^G(L)_\mathbf{r}$ has such a presentation if $L > 1$.*

5.2. **Counterexamples to Degree 1 generation.** Now we'll see how to generate examples of $(\mathbf{r}, \mathcal{T}, L)$ such that $S_\mathcal{T}^L(\mathbf{r})$ is not generated in degree 1. We will begin by defining a certain class of paths in the tree $\mathcal{T}$.

Definition 5.3. *Let $\mathcal{T}$ have an even number of leaves. Let $O(\mathcal{T})$ be the set of paths in $\mathcal{T}$ with the property that a weighting $\omega \in S_\mathcal{T}$ which assigns all odd numbers to elements of $V(\mathcal{T})$, weights the edges of any member of $O(\mathcal{T})$ with an odd number under the parity condition.*

Let us see that this is a well-defined set. It suffices to show that the parity of the members of $V(\mathcal{T})$ determines the parity of every edge in $\mathcal{T}$. This follows from induction on the number of edges in $\mathcal{T}$. To see that members of $O(\mathcal{T})$ are paths which never intersect, note that a lone odd number can never appear in a trinode,

nor can three odd numbers appear in a trinode. In particular, any pair of paired edges forms a member of $O(\mathcal{T})$.
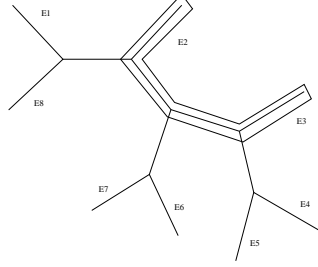

FIGURE 10. E2 and E3 are lone leaves connected by an element of $O(\mathcal{T})$

Proposition 5.4. *Let* $(\mathbf{r}, \mathcal{T}, L)$ *be such that the endpoints of each* $\gamma \in O(\mathcal{T})$ *are given the same parity, with some pair of endpoints* (E, F) *odd. Then if there is a degree* 2 *weighting which assigns* 0 *to any edge in* γ, $S_{\mathcal{T}}^{2L}(\mathbf{r})$ *is not generated in degree* 1.

Proof. All degree 1 elements must assign odd numbers to the edges on the path joining E and F. No two odd numbers add to 0. $\qquad\square$

Corollary 5.5. *The semigroup* $S_{\mathcal{T}}^{2L}(\vec{1})$ *is generated in degree* 1 *if and only if* $\mathcal{T}$ *has the property that no leaf is lone and* $L > 1$

Proof. The condition that the members of $\mathbf{r}$ sum to an even number forces us to only consider trees $\mathcal{T}$ with an even number of leaves. First we show that a tree with lone leaves has a degree 2 weighting satisfying the conditions of proposition 5.4. Since $L > 1$, it suffices to note that for any tree $\mathcal{T}$, and internal edge $e \in \mathcal{T}$, there is a weighting that assignes the edge e zero and every other edge 2. If $\mathcal{T}$ contains only paired leaves, we can restrict to the tree $c(\mathcal{T})$ and consider halved weightings without the parity condition. In this context, the weighting which assigns every edge 1 can be factored only if $L > 1$. This finishes the *only if* portion of the statement. The *if* portion of the statement is taken care of by Theorem 1.8. $\qquad\square$

Remark 5.6. *Trees with the property that no leaf is lone are called Good Trees in* [HMSV], *where they were introduced by Andrew Snowden for the purpose of proving the analogue of Corollary 5.5 for* $S_{\mathcal{T}}(\vec{1})$.

5.3. **The Case when L is odd.** When the level L is odd, the polytope $P_3(L)$ is no longer integral, however its Minkowski square $P_3(2L)$ is integral, so clearly there are elements of $P_3(2L)$ which cannot be integrally factored, specifically the corners. This observation has a generalization.

Definition 5.7. *Let* $IP_3(L)$ *be the convex hull of the integral points of* $P_3(L)$. *Let* Ω *be the set of elements in the graded semigroup of lattice points of* $P_3(L)$ *such that* $\frac{1}{deg(Q)}Q \in P_3(L) \setminus IP_3(L)$.

Let $(E, F, G) = Q \in P_3(L)$ be integral with L odd, and suppose E, F, or $G \geq \frac{L-1}{2} + 1$. Then, by the triangle inequalities we must have $F + G \geq \frac{L-1}{2} + 1$, so $E + F + G \geq L + 1$, a contradiction. This shows that $IP_3(L)$ is contained in the

intersection of $P_3(L)$ with the halfspaces $E, F, G \leq \frac{L-1}{2}$, this identifies $IP_3(L)$ as the convex hull of the set

$$\{(0,0,0), (\frac{L-1}{2}, \frac{L-1}{2}, 0), (\frac{L-1}{2}, 0, \frac{L-1}{2}), (0, \frac{L-1}{2}, \frac{L-1}{2}),$$
$$(\frac{L-1}{2}, \frac{L-1}{2}, 1), (\frac{L-1}{2}, 1, \frac{L-1}{2}), (1, \frac{L-1}{2}, \frac{L-1}{2})\}.$$
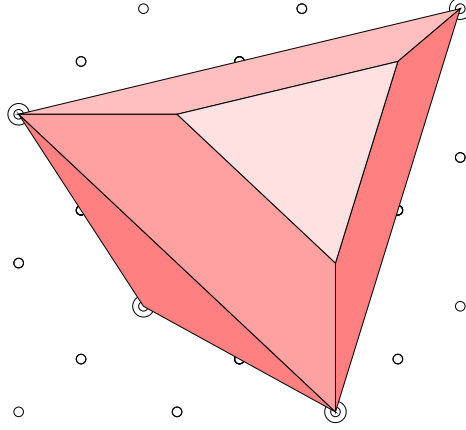
The case $IP_3(5)$ is pictured below.

FIGURE 11. The Polytope $IP_3(5)$

Proposition 5.8. *Any $Q \in \Omega$ cannot be integrally factored.*

Proof. This follows from the observation that if $Q = E_1 + \ldots + E_n$ then $\frac{1}{n}Q$ is in the convex hull of $\{E_1, \ldots, E_n\}$. □

A factorization of any element ω such that $i_v^*(\omega) = Q$ gives a factorization of Q. So any $\omega \in U_{c(\mathcal{T})}^L(\mathbf{r})$ with a $i_v^*(\omega) \in \Omega$ is necessarily an obstruction to generation in degree 1, this also turns out to be a sufficient obstruction criteria.
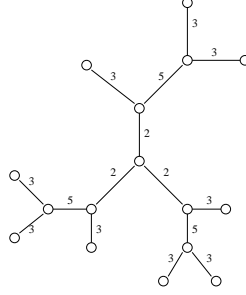
Theorem 5.9. *Let $\mathcal{T}$ and $\mathbf{r}$ satisfy the same conditions as admissibility, and let $L \neq 2$. Then $U_{c(\mathcal{T})}^L(\mathbf{r})$ is generated in degree 1 if and only if*

$$i_v^*(\omega) \in U_Y^L \setminus \Omega$$

for all $v \in I(c(\mathcal{T}))$, $\omega \in U_{c(\mathcal{T})}^L(\mathbf{r})$. In this case all relations are generated by those of degree at most 3.

Proof. We analyze $IP_3(L)$ in the same way we did $P_3(2L)$. The reader can verify that the integral points of $C(m_1, m_2, m_3) \cap P_3(L)$ are the same as the integral points of $C(m_1, m_2, m_3) \cap IP_3(L)$. The possibilities are represented by slicing the cubes in Figure 5 along the plane formed by the upper right or lower left collection of three non-filled dots, depending on the cube, and then restricting to the convex hull of the remaining integral points. All cases are lattice equivalent to one of the polytopes listed in Figure 4, after considering two and one dimensional cases as facets of neighboring three dimensional polytopes. Since any element of U_Y^L not in Ω is necessarily a lattice point of a Minkowski sum of $IP_3(L)$, the theorem follows by the same arguments used to prove Theorems 1.8 and 1.9 □

5.4. **Necessity of Degree** 3 **Relations.** Now we show that there are large classes of admissible $(\mathcal{T}, \mathbf{r}, L)$ which require degree 3 relations. We will exhibit a degree 3 weighting which has only two factorizations. The tree $\mathcal{T}$ with weight $\omega_{\mathcal{T}}$ is pictured below, it is an element of $S_{\mathcal{T}}(\vec{2})$. In all that follows all weightings are considered to have been halved.

FIGURE 12. $\omega_{\mathcal{T}}$

Notice that $\omega_{\mathcal{T}}$ has 3-way symmetry about the central trinode, we will exploit this by considering the tree $\mathcal{T}'$ with restricted weighting $\omega_{\mathcal{T}'}$ pictured in Figure 13. We find the weightings that serve as a degree 1 factors of $\omega_{\mathcal{T}'}$. First of all, any degree 1 weighting which divides $\omega_{\mathcal{T}'}$ must be as in Figure 14.
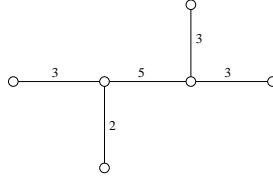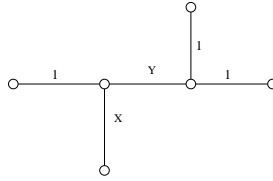

FIGURE 13. $\omega_{\mathcal{T}'}$

FIGURE 14.

It suffices to find the possible values of X and Y. Both must be ≤ 2, which shows that Y can be either 2 or 1. Now, by the triangle inequalities, any X paired with $Y = 1$ must be ≥ 1. Since two members of a factorization must have $Y = 2$, X must also have a value ≤ 1 on these factors. There are exactly two possibilities determined by the value of X, both are shown in Figure 15. Any factorization of $\omega_{\mathcal{T}}$ is determined by its values on the central trinode, and these values must be weights composed entirely of 0 and 1. There are exactly two such variations, making (of course) the Degenerated Segre Cubic.
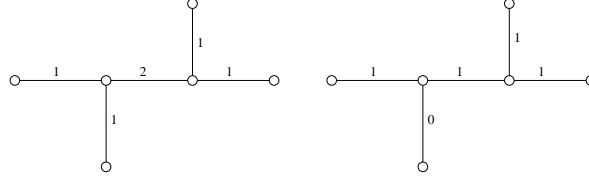
FIGURE 15.

We have not specified a level L for this weighting, but the same argument applies for any level large enough to admit $\omega_{\mathcal{T}}$ as a weighting in degree 3. For any tree $\mathcal{T}^*$, edge $e^* \in tree^*$, and weight $\omega_{\mathcal{T}^*}$ we can create a new weight on a larger tree by adding a vertex in the middle of e^*, attaching a new leaf edge at that vertex, and weighting the both sides of the split e^* with $\omega_{\mathcal{T}^*}(e^*)$, and the new edge with 0. Using this procedure on any $(\mathcal{T}^*, e^*, \omega_{\mathcal{T}^*})$, and $(\mathcal{T}, e, \omega_{\mathcal{T}})$ for any edge $e \in \mathcal{T}$, can create a new weighted tree by identifying the new 0-weighted edges. An example of this procedure, which we call merging, is pictured below. In this way many examples of unremoveable degree 3 relations can be made.
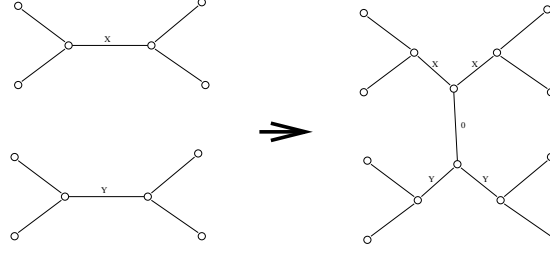

FIGURE 16. Merging two tree weightings.

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